3/4/26 Quiz!

No office hours Thurs (special seminar)

Last time:

- $\vec{v}_1, -, \vec{v}_m$ are <u>linearly indep</u> if none of them is redundant.

  Otherwise we say they are <u>linearly dependent</u>.

- Let V be a subspace of $\mathbb{R}^n$ and $\vec{v}_1, -, \vec{v}_m \in V$. We say $\vec{v}_1, -, \vec{v}_m$ <u>form a basis of V</u> if they span V and are lin. indep.

<u>Proposition</u> if $\vec{v}_1, -, \vec{v}_m$ form a basis of a subspace V of $\mathbb{R}^n$ then they span V but no subset spans V.

<u>proof</u>

They span V by defn. Suppose (for example) that $\vec{v}_1, -, \vec{v}_{m-1}$ span V.

Then $\exists\, a_1, -, a_{m-1}$ s.t. $a_1\vec{v}_1 + - + a_{m-1}\vec{v}_{m-1} = \vec{v}_m$ so $\vec{v}_m$ is redundant. This contradicts the assumption that $\vec{v}_1, -, \vec{v}_m$ form a basis. //

Let V be a subspace of $\mathbb{R}^n$ and let $\vec{v}_1, -, \vec{v}_m \in V$ such that we know they span V.

Let A be the matrix whose columns are $\vec{v}_1, -, \vec{v}_m$, so V is the col. space of A.

<u>Question</u> How do we tell if the vectors are lin indep or not (hence form a basis for V)?

Here's a "cheap" trick (Th 3.2.5), by example

<u>Ex</u>

$$\underset{\vec{v}_1}{\begin{bmatrix}1\\0\\2\\0\\3\end{bmatrix}} \quad \underset{\vec{v}_2}{\begin{bmatrix}2\\0\\3\\4\\5\end{bmatrix}} \quad \underset{\vec{v}_3}{\begin{bmatrix}3\\2\\1\\5\\6\end{bmatrix}}$$

Follow the 0's. $\vec{v}_2$ clearly not a mult of $\vec{v}_1$

$\vec{v}_3$ not a lin comb of $\vec{v}_1, \vec{v}_2$ b/c of 2^{nd} entry

So no redundant vectors.

But this usually isn't so useful.

Now follow up on previous remark.

<u>Def</u> Let $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ be vectors. An equation of the form

$$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0} \qquad (*)$$

is a <u>(linear) relation</u> among the $\vec{v}_i$.

<u>Note</u>: Taking $c_1 = \cdots = c_m = 0$ always works, so the important issue is whether there is a <u>nontrivial</u> choice that works.

<u>theorem</u> (3.2.7) (much more useful)

The vectors $\vec{v}_1, \ldots, \vec{v}_m$ are <u>lin dependent</u> iff the equation $(*)$ has a nontrivial solution. Equivalently

$\vec{v}_1, \ldots, \vec{v}_m$ are <u>lin. indep</u> iff $(*)$ has only the trivial solution we essentially proved this last time. (see book for full proof).

Here's a more general fact. Think of a set of vectors as columns of a matrix.

Thm (See Th 3.2.8 and Summary 3.2.9)

Let A be an $n \times m$ matrix. Then the following are equivalent.

(a) The cols are lin. indep.

(b) The equation (*) has only the trivial solution.

(c) $\ker(A) = \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^m$.

(d) $\operatorname{rank}(A) = m$

proof

It's enough to check

$$\begin{array}{ccc} (a) & \Rightarrow & (b) \\ \Uparrow & & \Downarrow \\ (d) & \Leftarrow & (c) \end{array}$$

$\boxed{(a) \Rightarrow (b)}$ comes from Th 3.2.7.

$\boxed{(b) \Rightarrow (c)}$

Note A defines $T: \mathbb{R}^m \to \mathbb{R}^n$. Let $\vec{v}_1, \dots, \vec{v}_m$ be the columns.

$\ker(A)$ = solution space of

$$A\vec{x} = \vec{0}$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$, $\vec{0} \in \mathbb{R}^n$.

from before

So $\vec{x} \in \ker(A)$ means

$$\begin{bmatrix} \vec{v}_1 & | & \cdots & | & \vec{v}_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \vec{0}$$

So $x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m = \vec{0}$. But (b) says then we must have

$x_1 = \cdots = x_m = 0$, which is (c). So (b) $\Rightarrow$ (c).